Yaglom-type limits for branching Brownian motion with absorption in the slightly subcritical regime

by Jiaqi Liu
University of Pennsylvania

Joint work with Julien Berestycki, Bastien Mallein, and Jason Schweinsberg

Outline

Definition of the process

Literature on the critical and subcritical cases

Yaglom-type limits in the slightly subcritical regime

Outline

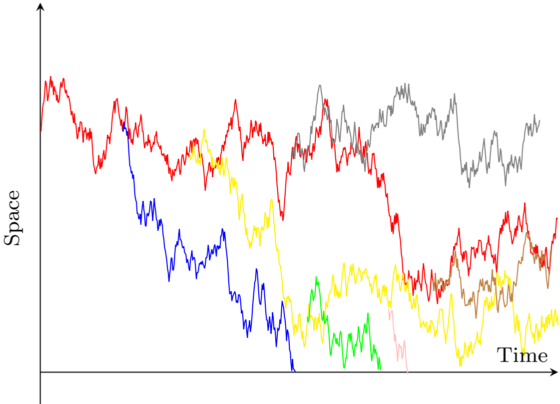
Definition of the process

Literature on the critical and subcritical cases

Yaglom-type limits in the slightly subcritical regime

BBM with absorption

- Started at $x \in \mathbb{R}_+$, a particle moves according to standard Brownian motion with drift $-\rho(\rho \in \mathbb{R})$ and is killed upon hitting the origin.
- With rate 1, this particle undergoes dyadic branching.
- Each offspring independently repeats above process and the system goes on.



Survival and extinction

Theorem (Kesten, 1978)

When $\rho \geq \sqrt{2}$, BBM with absorption dies out almost surely.
When $\rho < \sqrt{2}$, there is a positive probability of survival.

Henceforth, we will focus on the subcritical ($\rho > \sqrt{2}$) and critical ($\rho = \sqrt{2}$) cases.

Survival and extinction

Theorem (Kesten, 1978)

When $\rho \geq \sqrt{2}$, BBM with absorption dies out almost surely. When $\rho < \sqrt{2}$, there is a positive probability of survival.

Henceforth, we will focus on the subcritical ($\rho > \sqrt{2}$) and critical ($\rho = \sqrt{2}$) cases.

Questions

Let $\rho = \sqrt{2} + \varepsilon$ where $0 < \varepsilon < 1$. To understand the transition from the subcritical case to the critical case, we are interested in the long-run behavior of the process conditioned on survival (i.e. Yaglom-type limits) in the slightly subcritical regime where ε is sufficiently small.

Outline

Definition of the process

Literature on the critical and subcritical cases

Yaglom-type limits in the slightly subcritical regime

Critical case ($\rho = \sqrt{2}$): Survival probability starting from large x

Let $c = (3\pi^2)^{1/3}/\sqrt{2}$. For $t \geq 0$, define

$$L(t) = ct^{1/3}.$$

Building on earlier work of Kesten (1978) and Berestycki, Berestycki and Schweinsberg (2014), Maillard and Schweinsberg (2020) proved that

Theorem (Maillard-Schweinsberg, 2020)

There exists a function $\phi : \mathbb{R} \to (0,1)$ *satisfying* $\lim_{x\to\infty} \phi(x) = 0$ *and* $\lim_{x\to-\infty} \phi(x) = 1$ *such that for all* $x \in \mathbb{R}$,

$$\lim_{t\to\infty} P_{L(t)+x}\left(\zeta \leq t\right) = \phi(x).$$

Critical case ($\rho = \sqrt{2}$): Yaglom-type limits

- Kesten (1978) proved a Yaglom-type limit theorem for the number of particles conditioned on survival for a long time.
- Maillard and Schweinsberg (2020) gave the joint asymptotic distribution of the survival time ζ, the number of particles N_t and the position of the rightmost particle M_t conditioned on survival for a long time.

Theorem (Maillard-Schweinsberg, 2020)

Suppose the process starts from a single particle at $x > 0$. Let $V \sim Exp(1)$ and $c = (3\pi^2)^{1/3}/\sqrt{2}$. Conditional on $\zeta > t$, as $t \to \infty$,

$$\left(t^{-2/3}(\zeta - t), t^{-2/9}\log N_t, t^{-2/9} M_t\right) \Rightarrow \left(\frac{3}{2^{1/2}c}V, \frac{3^{1/3}c^{2/3}}{2^{1/6}}V^{1/3}, \frac{3^{1/3}c^{2/3}}{2^{2/3}}V^{1/3}\right).$$

Subcritical case ($\rho > \sqrt{2}$): Long run survival probability and Yaglom-type limits

Let P_x^ρ be the probability measure for BBM with absorption started from a single particle at $x > 0$ with drift $-\rho$.
Let E_x^ρ be the expectation under P_x^ρ.
Let N_t^ρ be the number of particles at time t under P_x^ρ.

Theorem (Harris-Harris, 2007)

For $\rho > \sqrt{2}$ and $x > 0$, there exists a constant K_ρ that is independent of x but dependent on ρ such that,

$$\lim_{t\to\infty} P_x^\rho(\zeta > t)\frac{\sqrt{2\pi t^3}}{x}e^{-\rho x+(\rho^2/2-1)t} = K_\rho$$

and

$$\lim_{t\to\infty} E_x^\rho[N_t^\rho|\zeta > t] = \frac{2}{\rho^2 K_\rho}.$$

Outline

Definition of the process

Literature on the critical and subcritical cases

Yaglom-type limits in the slightly subcritical regime

Existence of the Yaglom limit law

Let $\mathcal{N}_t^\rho$ be the set of particles alive at time t and $\{X_s(u), 0 \leq s \leq t\}_{u \in \mathcal{N}_t^\rho}$ be the past trajectories of particles alive at time t. Define $D_t = \sum_{u \in \mathcal{N}_t^\rho} \delta_{X_t(u)}$.

Proposition (Berestycki, L., Mallein, Schweinsberg (2022+))

For all $\rho > \sqrt{2}$ and $x > 0$, the probability distribution

$$\mathcal{D}^\rho(\cdot) = \lim_{t \to \infty} P_x^\rho(D_t \in \cdot | \zeta > t)$$

is well defined. Furthermore,

$$\mathcal{D}^\rho(A) = P_{\mathcal{D}^\rho}^\rho(D_t \in A | \zeta > t).$$

and $\mathcal{D}^\rho$ is the minimal quasi-stationary distribution for BBM with absorption and drift $-\rho$.

Survival probability starting from large x

For every $0 < \varepsilon < 1$, define

$$L_\varepsilon(t) = \varepsilon^{-1/2} F^{-1}(\varepsilon^{3/2} t),$$

where

$$F(u) = u - \omega \arctan(u/\omega), \qquad \omega = 2^{-3/4}\pi.$$

Theorem (Berestycki, L., Mallein, Schweinsberg (2022+))

Suppose there exists a positive constant C such that $0 < t_\varepsilon \varepsilon^{3/2} < C$ for all ε sufficiently small. Then for every $\delta > 0$, there exist postive constants C_1 and C_2 such that for all ε sufficiently small,

$$P^\rho_{L_\varepsilon(t_\varepsilon) - C_1}(\zeta > t_\varepsilon) < \delta,$$

and

$$P^\rho_{L_\varepsilon(t_\varepsilon) + C_2}(\zeta < t_\varepsilon) < \delta.$$

Properties of $L_\varepsilon(t)$

$$L_\varepsilon(t) = \varepsilon^{-1/2} F^{-1}(\varepsilon^{3/2} t),$$

where $F : u \mapsto u - \omega \arctan(u/\omega)$ and $\omega = 2^{-3/4}\pi$.

- If $t \ll \varepsilon^{-3/2}$, then for $c = (3\pi^2)^{1/3}/\sqrt{2}$,

  $$L_\varepsilon(t) = ct^{1/3} + o(1)$$

  which matches up with the critical case.
- If $t \gg \varepsilon^{-3/2}$, then $L_\varepsilon(t)$ behaves like a linear function

  $$L_\varepsilon(t) = \varepsilon t + \frac{\omega\pi}{2}\varepsilon^{-1/2} + o\left(\varepsilon^{-1/2}\right).$$

Yaglom-type limit for the expected number of particles

We show that the long-run expected number of particles conditioned on survival grows exponentially as $1/\sqrt{\varepsilon}$ as the process approaches criticality.

Theorem (L., 2020)

There exist positive constants C_1 and C_2 such that for ε small enough,

$$e^{C_1/\sqrt{\varepsilon}} \leq E[E^{\rho}_{\mathcal{D}^{\rho}}[N^{\rho}_0]] \leq e^{C_2/\sqrt{\varepsilon}}.$$

Equivalently, for ε small enough,

$$e^{C_1/\sqrt{\varepsilon}} \leq \lim_{t\to\infty} E^{\rho}_x[N^{\rho}_t|\zeta > t] \leq e^{C_2/\sqrt{\varepsilon}}.$$

Expected behavior vs. Typical behavior

If the process started from the Yaglom limit law, is the expected behavior of the process the same as its typical behavior?

Or in other words,

Conditioned on survival until an unusually large time t, does the long-run expected behavior of the process gives the long-run typical behavior?

Yaglom-type limits for the typical behavior

Let ζ be the extinction time, N_t^ρ be the number of particles at time t and M_t^ρ be the rightmost position at time t.

Theorem (Berestycki, L., Mallein, Schweinsberg (2022+))

Let V have an exponential distribution with mean one. For BBM with absorption and drift $-\rho$ started from the Yaglom limit $\mathcal{D}^\rho$, we have the joint convergence in distribution as $\varepsilon \to 0$

$$\left(\varepsilon\zeta, \varepsilon^{1/3}\log N_0^\rho, \varepsilon^{1/3}M_0^\rho\right) \Rightarrow \left(\frac{1}{\sqrt{2}}V, \frac{(3\pi^2)^{1/3}}{2^{1/6}}V^{1/3}, \frac{(3\pi^2)^{1/3}}{2^{2/3}}V^{1/3}\right).$$

Remark: As the process approaches criticality, conditioned on survival up to a large time t, the additional time for which the process will survive is $O(\varepsilon^{-1})$, the long-run number of particles grows exponentially as $\varepsilon^{-1/3}$, and the long-run rightmost position grows polynomially as $\varepsilon^{-1/3}$.

Expected behavior vs. Typical behavior

In the slightly subcritical regime, conditioned on survival up to time t, the typical number of survival particles is very different from its expected number,

$$\log N_t^\rho = O_p(\varepsilon^{-1/3}), \quad \log E_x^\rho[N_t^\rho] = O(\varepsilon^{-1/2}).$$

Expected behavior vs. Typical behavior

In the slightly subcritical regime, conditioned on survival up to time t, the typical number of survival particles is very different from its expected number,

$$\log N_t^\rho = O_p(\varepsilon^{-1/3}), \quad \log E_x^\rho[N_t^\rho] = O(\varepsilon^{-1/2}).$$

- The reason for this difference is that the mean is dominated by rare events in which the number of particles is unusually high.
- In the slightly subcritical regime, conditioned on $\zeta > t$, we have

$$N_t^\rho \approx \exp\left(\frac{(3\pi^2)^{1/3}}{2^{1/6}} V^{1/3} \varepsilon^{-1/3}\right)$$

- We have

$$E_x^\rho[N_t^\rho | \zeta > t] \approx \int_0^\infty \exp\left(\frac{(3\pi^2)^{1/3}}{2^{1/6}} s^{1/3} \varepsilon^{-1/3}\right) \cdot e^{-s} ds,$$

  and the integrand is maxmized when $s = O(\varepsilon^{-3/2})$.

A heuristic explanation

Recall that ζ is the survival time.

- Harris and Harris (2007) proved that

$$P_x^\rho(\zeta > t) \sim \frac{K_\rho}{\sqrt{2\pi t^3}} x e^{\rho x - \sqrt{2}\varepsilon t}.$$

- We get

$$\lim_{t\to\infty} P_x^\rho(\zeta > t+s | \zeta > t) = \lim_{t\to\infty} \frac{P_x^\rho(\zeta > t+s)}{P_x^\rho(\zeta > t)} = e^{-\sqrt{2}\varepsilon s}.$$

- Therefore conditioned on $\zeta > t$, $\zeta - t \sim Exp(\sqrt{2}\varepsilon)$.

A heuristic explanation

Recall that N_t^ρ is the number of particles at time t and M_t^ρ is the rightmost position at time t.

- Consider BBM with drift $-\rho = -\sqrt{2} - \varepsilon$ in the strip $[0, K]$. The density $p_t(x, y)$ for this process satisfies

$$p_t(x,y) \approx \frac{2}{K} e^{(1-\rho^2/2-\pi^2/2K^2)t} e^{\rho x} \sin\left(\frac{\pi x}{K}\right) e^{-\rho y} \sin\left(\frac{\pi y}{K}\right).$$

- The density $p_t(x, y)$ gives a good approximation of the particle configuration if K is large enough that particles are unlikely to be killed at K, but small enough that the truncated second moment is comparable with the first moment. A good choice of K would be near the rightmost position.
- Particles eventually settle into a configuration in which the density of particles near y is proportional to $e^{-\rho y} \sin(\pi y/K)$. Therefore, the probability that a particle is close to K is approximately $e^{-\rho K}$, which implies that $\log N_t^\rho \approx \rho K \approx \rho M_t^\rho$.

A heuristic explanation

- In a typical realization, conditioned on $\zeta > t$,

$$\zeta - t \approx \frac{1}{\sqrt{2}} V \varepsilon^{-1} = O_p(\varepsilon^{-1})$$

and the process should behave very much like the critical process.

- Let $c = (3\pi^2)^{1/3}/\sqrt{2}$. In the critical case, if the process survives until time $t+s$, the position of the rightmost particle at time t should be around $cs^{1/3}$ and the number of particles at time t should be near $e^{\sqrt{2}cs^{1/3}}$.
- Conditioned on $\zeta > t$, we expect

$$M_t^\rho \approx c(\zeta - t)^{1/3} \approx c\left(\frac{1}{\sqrt{2}} V \varepsilon^{-1}\right)^{1/3} = \frac{(3\pi^2)^{1/3}}{2^{2/3}} V^{1/3} \varepsilon^{-1/3},$$

$$\log N_t^\rho \approx \sqrt{2}c(\zeta - t)^{1/3} \approx \sqrt{2}c\left(\frac{1}{\sqrt{2}} V \varepsilon^{-1}\right)^{1/3} = \frac{(3\pi^2)^{1/3}}{2^{1/6}} V^{1/3} \varepsilon^{-1/3}.$$

Properties of $L_\varepsilon(t)$

$$L_\varepsilon(t) = \varepsilon^{-1/2} F^{-1}(\varepsilon^{3/2} t),$$

where $F : u \mapsto u - \omega \arctan(u/\omega)$ and $\omega = 2^{-3/4}\pi$.

- If $t \ll \varepsilon^{-3/2}$, then for $c = (3\pi^2)^{1/3}/\sqrt{2}$,

  $$L_\varepsilon(t) = ct^{1/3} + o(1)$$

  which matches up with the critical case.
- If $t \gg \varepsilon^{-3/2}$, then $L_\varepsilon(t)$ behaves like a linear function

  $$L_\varepsilon(t) = \varepsilon t + \frac{\omega\pi}{2}\varepsilon^{-1/2} + o\left(\varepsilon^{-1/2}\right).$$

Derivation of $L_\varepsilon(t)$

- Recall that for BBM with drift $-\rho = -\sqrt{2} - \varepsilon$ in the strip $[0, K]$, the density is

$$p_t(x,y) \approx \frac{2}{K} e^{(1-\rho^2/2-\pi^2/2K^2)t} e^{\rho x} \sin\left(\frac{\pi x}{K}\right) e^{-\rho y} \sin\left(\frac{\pi y}{K}\right),$$

which implies that

$$\frac{d}{ds} N_s^\rho = \left(1 - \frac{\rho^2}{2} - \frac{\pi^2}{2K^2}\right) N_s^\rho \approx \left(-\sqrt{2}\varepsilon - \frac{\pi^2}{2K^2}\right) N_s^\rho.$$

- Fix $t > 0$. We think of K as a function of s. Note that $K(s) \approx \log N_s^\rho / \rho$. In the slightly subcritical regime, $K(s)$ roughly satisfies

$$\frac{d}{ds} K(s) = \frac{1}{\rho N_s^\rho} \frac{d}{ds} N_s^\rho \approx \left(-\varepsilon - \frac{\pi^2}{2\sqrt{2}}\right) \frac{1}{K(s)^2}.$$

- Note that $L_\varepsilon(t) = K(0)$ with $K(t) = 0$. Letting $\omega = 2^{-3/4}\pi$, we get an implicit expression for $L_\varepsilon(t)$

$$L_\varepsilon(t) = \varepsilon t + \omega \varepsilon^{-1/2} \arctan\left(\frac{\varepsilon^{1/2} L_\varepsilon(t)}{\omega}\right).$$